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APPROXIMATE INTERVALS AND TESTS FOR PRODUCTS AND RATIOS OF BINOMIAL PROBABILITIES

by

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Technical Report No. 89
Department of Statistics ONR Contract

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APPROXIMATE INTERVALS AND TESTS FOR PRODUCTS

AND RATIOS OF BINOMIAL PROBABILITIES

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ABSTRACT

The data are the observed numbers of successes for sets of independent trials from binomial distributions (success and failure are the possible outcomes for each binomial event). Desired are confidence intervals and tests for the product of the success probabilities for these binomial distributions. Also desired are intervals and tests for the product of a specified set of these probabilities divided by the product of the remaining probabilities. The situation of interest is that where, for all distributions, the success probabilities are of at least moderate size (say, at least 2/3) and the numbers of trials are at least moderately large. Approximate intervals and tests are developed by use of logarithms, expansions, and normal approximations. These results have application in investigating and comparing system reliabilities.

*Based on work performed for the Quality Evaluation Laboratory, U. S. Naval Torpedo Station, Keyport, Washington.

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INTRODUCTION AND RESULTS

Consider independent data from N binomial distributions of success and failure, where $N \geq 2$. The i-th distribution has unknown probability p_i for a success, involves n_i trials, and its observed fraction of successes is denoted by $p_i'(i=1,\ldots,N)$. Confidence intervals and significance tests are desired for $\prod_{i=1}^{N} p_i$ and for

$$\left(\prod_{i=1}^{m} p_i\right)\Big/\left(\prod_{i=m+1}^{N} p_i\right),$$

where m is specified and such that $1 \leq m \leq N - 1$. Of course, by suitable indexing, any m of the binomial probabilities could be $p_1,\ldots,p_m$.

Investigation of such products of probabilities can be helpful in studies of system reliability, where the ratio occurs when a comparison of the reliabilities of two systems is of interest. Here, a success could represent satisfactory performance of a subsystem while failure represents unsatisfactory performance. Very frequently, all the p_i are at least 2/3 for reliability situations. Otherwise, unacceptable system reliabilities would occur.

For brevity only confidence intervals are explicitly considered. Significance tests are obtainable from the intervals in the usual manner. The null value investigated depends on the circumstances. Often, however, the null value for the ratio will be unity (for example, when unit ratio represents equal reliability for two systems).

The case where $\prod_{i=1}^{N} p_i$ is investigated is considered first. The basic statistic for this case is

$$T_p = \log_e\left[\prod_{i=1}^{N} p_i'\right] + \sum_{i=1}^{N} (1-p_i')/2n_ip_i'.$$

When all $n_ip_i' \geq 1$, all $p_i \geq 2/3$, all $n_ip_i \geq 10$, and $n_i(1-p_i) \geq 5$ for at least two values of i, the distribution of T_p is, approximately, normal with mean $\log_e\left[\prod_{i=1}^{N} p_i\right]$ and variance

$$\sum_{i=1}^{N} (1-p_i)/n_ip_i, \tag{1}$$

which is estimated by

$$s^2 = \sum_{i=1}^{N} (1-p_i')/n_ip_i'.$$

Thus, approximately,

$$\left\{T_p - \log_e\left[\prod_{i=1}^{N} p_i\right]\right\}s^{-1} \tag{2}$$

has a standardized normal distribution (zero mean and unit variance).

One-sided and two-sided confidence intervals are easily obtained by use of (2). For example,

$$P\{\exp[T_p-sK_{\alpha/2}] \leq \prod_{i=1}^{N} p_i \leq \exp[T_p+sK_{\alpha/2}]\}$$

$$\doteq 1 - \alpha,$$

where $K_{\alpha/2}$ is the standardized normal deviate exceeded with probability $\alpha/2$, defines an equal-tail confidence interval for $\prod_{i=1}^{N} p_i$ whose confidence coefficient is approximately $1 - \alpha$.

Now, consider the case where a ratio is investigated. The basic statistic for this case is

$$T_R = \log_e\left[\prod_{i=1}^{N} p_i'\right] - \log_e\left[\prod_{i=m+1}^{N} p_i'\right]$$

$$+ \sum_{i=1}^{m} (1-p_i')/2n_ip_i' - \sum_{i=m+1}^{N} (1-p_i')/2n_ip_i'.$$

When all $n_ip_i' \geq 1$, all $p_i \geq 2/3$, all $n_ip_i \geq 10$, and $n_i(1-p_i) \geq 5$ for at least two values of i, the distribution of T_R is, approximately, normal with mean

$$\log_e\left[\left(\prod_{i=1}^{m} p_i\right)\Big/\left(\prod_{i=m+1}^{N} p_i\right)\right]$$

and the variance expression (1). Since (1) is estimated by s^2,

$$\left\{T_R - \log_e\left[\left(\prod_{i=1}^{m} p_i\right)\Big/\left(\prod_{i=m+1}^{N} p_i\right)\right]\right\} s^{-1}$$

has a standardized normal distribution, approximately, and provides the basis for obtaining one-sided and two-sided confidence intervals for the ratio.

In general, let the value of a confidence coefficient be represented as $1 - \alpha$ (then α is the significance level of the corresponding test). The accuracy of a confidence coefficient tends to decrease as α decreases. That is, the normal approximation tends to become less accurate as values farther and farther into the tails are considered. Use of $\alpha \geq .001$ should be satisfactory if all the n_i are at least moderately large (say, all $n_i \geq 50$). In any case, when the other conditions hold, use of $\alpha \geq .01$ should be satisfactory. Equal-tail results tend to have the most accurately determined probabilities (for example, see ref. 1) and, in general, use of $\alpha \geq .005$ should be satisfactory.

The final section contains verification of the stated results, including a discussion of the conditions on the n_i, p_i', and p_i.

VERIFICATION

Consider $\log_e p_i'$, where $p_i' > 0$. This can be expressed as

$$\log_e(p_i' - p_i + p_i) = \log_e\{p_i[1 + (p_i' - p_i)/p_i]\}$$
$$= \log_e p_i + \log_e[1 + (p_i' - p_i)/p_i].$$

Thus,

$$t_i = \log_e p_i' + (1 - p_i')/2n_i p_i'$$
$$= \log_e p_i + (p_i' - p_i)/p_i + (1 - p_i')/2n_i p_i' - (p_i' - p_i)^2/2p_i^2$$
$$+ (p_i' - p_i)^3/3p_i^3 - (p_i' - p_i)^4/4p_i^4 + \ldots$$

defines t_i and

$$E(t_i) = \log_e p_i + O(n_i^{-3/2}).$$

Also, the variance of t_i is

$$(1 - p_i)/n_i p_i + O(n_i^{-2}),$$

and the third moment about the mean, when divided by the 3/2 power of the variance, equals

$$(1 - 2p_i)/[n_i p_i(1 - p_i)]^{1/2} + O(n_i^{-3/2}).$$

Moreover, the fourth moment about the mean, when divided by the square of the variance, equals $3 + O(n_i^{-1})$.

These moment values, in combination with the form of the expansion, seem to imply that the distribution of t_i is approximately normal when $n_i p_i' \geq 1$, $p_i \geq 1/2$, $n_i p_i \geq 10$, and $n_i(1 - p_i) \geq 5$. This is evidently the case (for example, see page 87 of ref. 2) when all the terms in the expression for $t_i - \log_e p_i$ except $(p_i' - p_i)/p_i$ are unimportant, which occurs when n_i is at least moderately large (say, $n_i \geq 15$). Of course, $n_i(1 - p_i) \geq 5$ implies that $n_i \geq 15$. The requirement $n_i p_i' \geq 1$ should not have much conditional effect on the distribution of t_i, since the probability of $n_i p_i' \geq 1$ is very nearly unity when $n_i p_i \geq 10$.

At least rough normality of t_i should occur when $n_i p_i' \geq 1$, $p_i \geq 2/3$, $n_i p_i \geq 10$, and $n_i(1 - p_i) \geq 2$. Addition of such t_i to at least two others that satisfy $n_i p_i(1 - p_i) \geq 5$ should yield a statistic whose distribution is nearly normal. In fact, the variance of a t_i for which $n_i p_i(1 - p_i) = 5$ is definitely greater than the variance of a t_i for which $n_i p_i(1 - p_i) = 2$, and can be much greater. Again, the requirement $np_i' \geq 1$ should have an unimportant conditional effect.

The remaining possibility is t_i for which $n_i p_i' \geq 1$, $p_i \geq 2/3$, $n_i p_i \geq 10$, and $n_i(1 - p_i) < 2$. Here, the variance of t_i decreases as $n_i(1 - p_i)$ decreases and is very definitely smaller than that for a t_i where $n_i(1 - p_i) = 5$. Also, t_i tends to have a unimodal distribution. Addition of a t_i with $n_i(1 - p_i) < 2$ to at least two t_i with $n_i(1 - p_i) \geq 5$ should yield a statistic that is very nearly normal. The effect of $n_i p_i' \geq 1$ is unimportant.

Since T_P is the summation of the t_i, and T_R is the summation of

$t_1, \ldots, t_m$ plus the summation of $-t_{m+1}, \ldots, -t_N$, both T_P and T_R should have distributions that are approximately normal. The above analysis motivates this statement for small N. This analysis in combination with the Central Limit Theorem provides the motivation for moderate and large values of N. That is, even if the sum of the variances for t_i with $n_i(1 - p_i) < 2$ is at least equal to the sum of the variances for the other t_i, the value of N becomes so large that the Central Limit Theorem should imply approximate normality for T_P and T_R. Approximate normality should always occur when the sum of the variances of the t_i with $n_i(1 - p_i) < 2$ is definitely smaller than the sum of the variances for the remaining t_i.

Finally, consider the motivation for including terms of the type $(1 - p_i')/2n_i p_i'$ in the expressions for T_P and T_R. The expectation of this term is $O(n_i^{-1})$ and the standard deviation of $\log_e p_i' + (1 - p_i')/2n_i p_i'$ is $O(n_i^{-1/2})$. This would seem to imply that terms of this type can be neglected. However, the values of p_i and n_i can vary substantially. Moreover, many terms can be added to obtain T_P and T_R. These various possibilities suggested that elimination of expectations that are $O(n_i^{-1})$ is advisable. Inclusion of the $(1 - p_i')/2n_i p_i'$ terms accomplishes this elimination.

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